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ON THE QUADRATIC MEAN RADIUS OF A POLYNOMIAL IN C(Z)  
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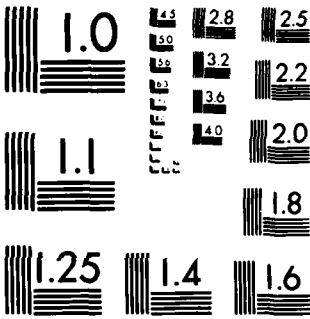
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ON THE QUADRATIC MEAN RADIUS  
OF A POLYNOMIAL IN  $C(z)$

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ON THE QUADRATIC MEAN RADIUS OF A POLYNOMIAL IN  $C[z]$

I. J. Schoenberg

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ABSTRACT

Let

$$P_n(z) = z^n + a_2 z^{n-2} + \dots + a_n = \prod_{j=1}^n (z - z_j)$$

be a polynomial in  $C[z]$  having the origin as the center of gravity of its zeros  $z_j$ . We call

$$R(P_n) = \left( \frac{1}{n} \sum_{j=1}^n |z_j|^2 \right)^{1/2}$$

the quadratic radius of  $P_n(z)$ . We also consider the derivative

$$P'_n(z) = nz^{n-1} + (n-2)a_2 z^{n-3} + \dots + a_{n-1} = n \prod_{k=1}^{n-1} (z - w_k)$$

and its quadratic radius

$$R(P'_n) = \left( \frac{1}{n-1} \sum_{k=1}^{n-1} |w_k|^2 \right)^{1/2}.$$

The main purpose of this note is to state

Conjecture 1. We have the inequality

$$(1) \quad R(P'_n) \leq \sqrt{\frac{n-2}{n-1}} R(P_n),$$

with the equality sign if and only all the zeros  $z_j$  of  $P_n(z)$  are real, or equivalently, all  $z_j$  are on a straight line of  $C$ .

We prove (1) for  $n = 3$ . Also for binomial polynomials of the form  $P_n(z) = z^n + a_k z^{n-k}$  ( $2 \leq k \leq n$ ). We prove directly other consequences of Conjecture 1.

AMS (MOS) Subject Classifications: 30C10, 30C15

Key Words: Zeros of polynomials in  $C[z]$ , Analogue of Rolle's theorem.

Work Unit Number 3 - Numerical Analysis and Scientific Computing

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### SIGNIFICANCE AND EXPLANATION

The main purpose of this note is to state a conjecture which may be regarded as an analogue of Rolle's theorem for arbitrary polynomials  $P_n(z)$  with real or complex coefficient, having the origin  $z = 0$  as the centroid of the zeros of  $P_n(z)$ . How the zeros  $z_j$  of  $P_n(z)$  crowd around their centroid is measured by the quadratic mean

$$R(P_n) = \left( \frac{1}{n} \sum_{j=1}^n |z_j|^2 \right)^{1/2}$$

of their distances from their centroid 0. If

$$R(P'_n) = \left( \frac{1}{n-1} \sum_{k=1}^{n-1} |w_k|^2 \right)^{1/2}$$

is the similar quantity for the derivative  $P'_n(z)$  having the zeros  $w_1, \dots, w_{n-1}$ , it is conjectured that

$$R(P'_n) \leq \sqrt{\frac{n-2}{n-1}} R(P_n) ,$$

with the equality sign if and only if the zeros  $z_j$  are all real, or equivalently, all  $z_j$  are on a straight line.

This is proved if  $n = 3$  and also for binomial polynomials.

ON THE QUADRATIC MEAN RADIUS OF A POLYNOMIAL IN  $C[z]$

I. J. Schoenberg

Introduction. Let  $c_1, c_2, \dots, c_n$  be  $n$  non-negative reals. We call the quadratic mean of the  $c_j$  the quantity  $M(c_j)$  defined by

$$(1) \quad M(c_j) = \left( \frac{1}{n} \sum_{j=1}^n c_j^2 \right)^{1/2}.$$

Moreover, let

$$(2) \quad P_n(z) = z^n + a_2 z^{n-2} + \dots + a_n, \quad (n \geq 2),$$

be a polynomial with real or complex coefficient with  $a_1 = 0$ . If

$$(3) \quad P_n(z) = \prod_{j=1}^n (z - z_j)$$

exhibits its zeros  $z_j$ , then clearly

$$(4) \quad z_1 + z_2 + \dots + z_n = 0,$$

which implies that in the complex plane the origin  $O$  is the center of gravity of the  $z_j$ . We define the quadratic mean radius of  $P_n$  by

$$(5) \quad R(P_n) = M(|z_j|) = \left( \frac{1}{n} \sum_{j=1}^n |z_j|^2 \right)^{1/2}.$$

We are here concerned with the effect of the operation of differentiation of  $P_n(z)$  on its  $R(P_n)$ , which we also simply call the quadratic radius of  $P_n$ . Concerning this effect we have a conjecture. We consider the derivative

$$(6) \quad P'_n(z) = nz^{n-1} + (n-2)a_2 z^{n-3} + \dots + a_{n-1} = n \prod_{k=1}^{n-1} (z - w_k),$$

having the zeros  $w_k$ , and we wish to compare the quadratic radii  $R(P_n)$  and  $R(P'_n)$ , looking for some analogue of Rolle's theorem.

The main purpose of the present note is to state

Conjecture 1. The quadratic radii  $R(P_n)$  and  $R(P'_n)$  satisfy the inequality

$$(7) \quad R(P'_n) \leq \sqrt{\frac{n-2}{n-1}} \cdot R(P_n),$$

with the equality sign if and only if all the zeros  $z_j$  of  $P_n(z)$  are real or, equivalently, all zeros  $z_j$  are on a straight line of  $C$ .

A trivial example

$$R(z^{n-1}) = 1 , \quad R(nz^{n-1}) = 0 .$$

The necessity of the equation

$$(8) \quad \frac{1}{n-1} \sum_{k=1}^{n-1} |w_k|^2 = \frac{n-2}{n-1} \left( \frac{1}{n} \sum_{j=1}^n |z_j|^2 \right)$$

for the reality of all zeros of  $P_n(z)$  is quite trivial, and this is the original source of the conjecture: As in this case the  $z_j$  and  $w_k$  are all real, we can remove in (8) all the absolute value signs. Moreover,

$$(9) \quad w_1 + w_2 + \dots + w_{n-1} = 0 .$$

From (4), (9), (2), and (6) we have

$$(10) \quad \begin{aligned} \sum_{j=1}^n z_j^2 &= -2 \sum_{j < j'} z_j z_{j'} = -2a_2 , \\ \sum_{k=1}^{n-1} w_k^2 &= -2 \sum_{k < k'} w_k w_{k'} = -\frac{n-2}{n} \cdot 2a_2 , \end{aligned}$$

and eliminating  $a_2$  between these equations, we obtain

$$(11) \quad \frac{1}{n-1} \sum_{k=1}^{n-1} w_k^2 = \frac{n-2}{n-1} \frac{1}{n} \sum_{j=1}^n z_j^2 ,$$

which is equivalent with (8), as the  $z_j$  and  $w_k$  are all real.

In the sequel we establish Conjecture 1 for the case where  $n = 3$ , and we derive some consequences of Conjecture 1, which are of course, only conjectured. In §2 we establish Conjecture 1 for binomial polynomials.

Fred W. Sauer, of the MRC Computing Staff, has verified the strict inequality (7) for some 25 numerically given complex  $P_n(z)$ , and in the last §5 we record three of his examples. For this I am very much obliged to Fred. I am also much impressed by the speed and precision of the Jenkins-Traub algorithm used in solving the equations  $P_n(z) = 0$  and  $P'_n(z) = 0$ .

1. Proof of Conjecture 1 when  $n = 3$ . Let

$$(1.1) \quad T = (z_1, z_2, z_3)$$

be the triangle of the complex plane having the zeros of  $P_3(z)$  as vertices. By our assumption (4) the centroid of  $T$  is in the origin  $O$ . We are now using the following theorem of van den Berg ([1], or [3, Chap. 7]).

Theorem 1 (van den Berg). Let  $E$  be the Steiner ellipse of the triangle  $T$ . This is the ellipse which is inscribed in  $T$  such that  $E$  is tangent to the sides of  $T$  in their midpoints. Then the zeros of  $P'_3(z)$  are identical with the foci  $w_1$  and  $w_2$  of the ellipse  $E$ . (See Fig. 1).

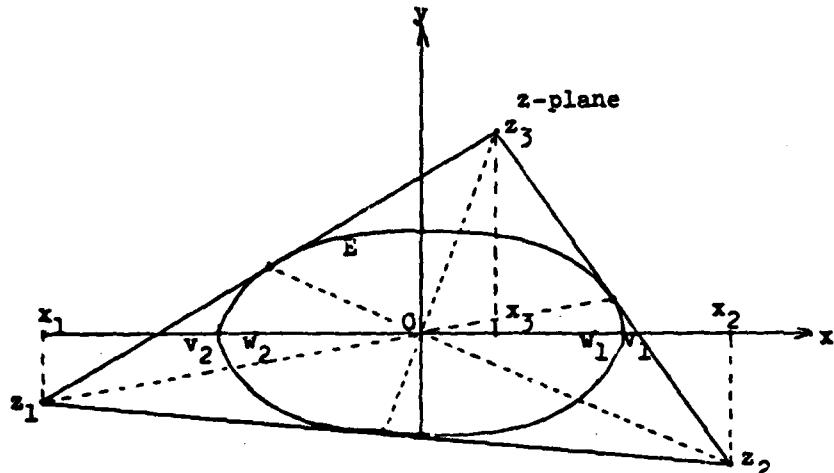


Fig.1

With  $z = x + iy$  and  $z_j = x_j + iy_j$  we place  $T \cup E$  so that the major axis  $v_1v_2$  of  $E$  is on the real  $x$ -axis, its center being at the origin. Let  $a$  and  $b$  be the semi-axes of  $E$  and  $a^2 - b^2 = c^2 = w_1^2 = w_2^2$ . We now subject the plane to the affine transformation

$$\begin{aligned} x(t) &= x \\ A_t : \quad y(t) &= ty, \quad (0 \leq t \leq 1), \end{aligned}$$

which contracts  $E$  toward the  $x$ -axis. As the semi-axes of  $E(t) = A_t E$  are  $a$  and  $bt$ ,

we find that the foci of  $E(t)$ , which we denote by  $w_1(t)$  and  $w_2(t)$ , are

$$w_1(t) = \sqrt{a^2 - b^2 t^2} + v_1, \text{ as } t \rightarrow 0,$$

and similarly  $w_2(t) \rightarrow -v_2$ . This shows, by continuity as  $t \rightarrow +0$ , that  $3(z-v_1)(z-v_2)$  is the derivative of the real cubic

$$(1.2) \quad (z-x_1)(z-x_2)(z-x_3).$$

By (10) we have  $(v_1^2 + v_2^2)/2 = (x_1^2 + x_2^2 + x_3^2)/6$ , and therefore

$$(1.3) \quad \frac{1}{6} \sum_1^3 |z_j|^2 \geq \frac{1}{6} \sum_1^3 x_j^2 = \frac{1}{2} (v_1^2 + v_2^2) = a^2 \geq \frac{1}{2} (|v_1|^2 + |v_2|^2).$$

A comparison of the extreme terms of (1.3) gives the desired inequality (7) which, for  $n = 3$ , is

$$(1.4) \quad R(P_3) \leq \sqrt{\frac{1}{2}} R(P_3).$$

Finally, the equality sign between the extreme terms of (1.3) implies that  $\sum |z_j|^2 = \sum x_j^2$ , and therefore  $z_j = x_j$  ( $j = 1, 2, 3$ ), so that the zeros of  $P_3(z)$  are real.

2. Verifying Conjecture 1 for binomial polynomials. There is a simple case when the quadratic radii  $R(P_n)$  and  $R(P'_n)$  can both be evaluated very simply explicitly. This is the case when the polynomial (1) is binomial, i.e. of the form  $P_n(z) = z^n + a_k z^{n-k}$  with  $2 \leq k \leq n$ . Without loss of generality we may assume that  $a_k = 1$ , hence

$$(2.1) \quad P_n(z) = z^n + z^{n-k}, \quad (2 \leq k \leq n),$$

whence

$$P'_n(z) = nz^{n-1} + (n-k)z^{n-k-1}.$$

We find for  $P_n(z) = z^{n-k}(z^k + 1)$  and, by (5), that

$$(2.2) \quad (R(P_n))^2 \equiv \frac{k}{n}.$$

Likewise  $P'_n(z) = nz^{n-k-1}(z^k + (n-k)/n)$ , and therefore, by (5), that

$$(2.3) \quad (R(P'_n))^2 = \frac{k}{n-1} \left( \frac{n-k}{n} \right)^2 / k.$$

The desired inequality (7) now easily reduces to proving

Lemma 1. We have the inequality

$$(2.4) \quad (n-2)^k \geq n^{k-2}(n-k)^2 \text{ if } 2 \leq k \leq n.$$

Proof: This is trivial if  $k = 2$ , or if  $k = n$ , and we may therefore assume that

$$(2.5) \quad 2 < k < n$$

and prove

Lemma 1'. We have the inequality

$$(2.6) \quad (n - 2)^k > n^{k-2}(n - k)^2 \text{ if } 2 < k < n .$$

This we will derive from the more general

Lemma 2. Let the reals  $x_1, \dots, x_k$ , not all equal to each other, have the mean

$$(2.7) \quad a = \frac{1}{k} \sum_1^k x_j ,$$

then

$$(2.8) \quad (x - a)^k > \prod_1^k (x - x_j) \text{ if } x > x_j (j = 1, \dots, k) .$$

Proof of Lemma 2. Taking logarithms, (2.8) is equivalent to

$$(2.9) \quad \log(x - a) > \frac{1}{k} \sum_1^k \log(x - x_j) \text{ if } x > \max x_j .$$

From (2.7) we have

$$x - a = \frac{1}{k} \sum_1^k (x - x_j)$$

and (2.9) amounts to

$$(2.10) \quad \log\left(\frac{1}{k} \sum_1^k (x - x_j)\right) > \frac{1}{k} \sum_1^k \log(x - x_j)$$

which follows from the strict concavity of  $\log x$  in  $0 < x < \infty$ . Indeed, for any strictly concave function  $f(x)$  in  $(0, \infty)$ , and for positive quantities

$$p_j = x - x_j, \text{ not all equal to each other ,}$$

we have the well known inequality

$$f\left(\frac{1}{k} \sum_1^k p_j\right) > \frac{1}{k} \sum_1^k f(p_j) .$$

For  $f(x) = \log x$  this amounts to (2.10).

Proof of Lemma 1'. We specialize Lemma 2 by choosing

$$x = x = \dots = x_{k-2} = 0, x_{k-1} = x_k = k, \text{ and } x = n .$$

For the mean value (2.7) we find

$$a = \frac{1}{k} \sum_1^k x_j = \frac{1}{k} \cdot 2k = 2 ,$$

while

$$\prod_{j=1}^{k-2} (x - x_j) = n^{k-2}, \quad (x - x_{k-1})(x - x_k) = (n - k)^2 .$$

Now (2.8) goes over into the desired inequality (2.6).

3. On Appell sequences: A corollary of Conjecture 1. We start from  $P_0(z) = 1$  and integrate it successively with arbitrary constants of integration, obtaining a so-called Appell sequence of polynomials

$$(3.1) \quad P_n(z) = \frac{1}{n!} (z^n + {}^n_1 c_1 z^{n-1} + {}^n_2 c_2 z^{n-2} + \dots + c_n), \quad (n = 1, 2, \dots) .$$

Here

$$(3.2) \quad (c_n), \quad (c_0 = 1; n = 1, 2, \dots)$$

is an arbitrarily prescribed sequence of real or complex constants. Evidently

$$(3.3) \quad P_n'(z) = P_{n-1}(z), \quad (n = 1, 2, \dots) .$$

Conversely, (3.3) and  $P_0(z) = 1$  imply (3.1).

Let

$$(3.4) \quad P_n(z) = \frac{1}{n!} \prod_{j=1}^n (z - z_j^{(n)})$$

describe the zeros of these polynomials. Without loss of generality we may assume that

$$(3.5) \quad c_1 = 0 ,$$

which implies that

$$(3.6) \quad \sum_{j=1}^n z_j^{(n)} = 0, \quad (n = 1, 2, \dots) .$$

Because of the relation (3.3) we wish now to apply Conjecture 1, hence the inequality (7), to any two consecutive pair of polynomials of the sequence (3.1). Of course, the results will only be conjectured, as Conjecture 1 has not been established. Since (7) may be written as

$$\frac{1}{n-1} \sum_{k=1}^{n-1} |w_k|^2 \leq \frac{n-2}{n-1} \frac{1}{n} \sum_{j=1}^n |z_j|^2 ,$$

or

$$\frac{1}{(n-1)(n-2)} \sum_{k=1}^{n-1} |w_k|^2 \leq \frac{1}{n(n-1)} \sum_{j=1}^n |z_j|^2 .$$

we obtain for the pair  $P_k(z), P_{k+1}(z)$ , with the notations of (3.4), the inequality

$$\frac{1}{k(k-1)} \sum_1^k |z_j^{(k)}|^2 \leq \frac{1}{(k+1)k} \sum_1^{k+1} |z_j^{(k+1)}|^2 .$$

By iterating this recurrent inequality we obtain

$$\frac{1}{k(k-1)} \sum_1^k |z_j^{(k)}|^2 \leq \frac{1}{n(n-1)} \sum_1^n |z_j^{(n)}|^2 \text{ for } n > k \geq 2 .$$

This we may write in the final form

$$(3.7) \quad (R(P_n))^2 \geq \frac{1}{k-1} (R(P_k))^2 \cdot (n-1) \quad (n > k \geq 2) .$$

An immediate (conjectured) consequence is the

Conjectured Corollary 1. If the polynomials (3.1) are

$$(3.8) \quad P_n(z) = z^n/n! \text{ for } n = 1, 2, \dots ,$$

then clearly

$$(3.9) \quad R(P_n) = 0 \text{ for all } n .$$

Let us now assume that

$$(3.10) \quad P_n(z) = z^n/n! \text{ for } n = 1, 2, \dots, k-1 \quad (k \geq 2) ,$$

while

$$(3.11) \quad P_k(z) = \frac{z^k - a^k}{k!}, \quad (a \neq 0) ,$$

then

$$(3.12) \quad R(P_n) \geq C\sqrt{n-1}, \quad \text{where } C = |a|/(k-1)^{1/2}, \quad \text{for } n > k .$$

Indeed, if in (3.7) we choose for  $k$  its value that appears in (3.11) we obtain the lower estimate (3.12), because  $R(P_k) = |a|$ .

4. A direct proof of (3.12) if  $k=2$ . Our belief in the truth of a conjecture is strengthened if we can prove directly a consequence of the conjecture. This is immediate: If  $k=2$  in (3.11), then in (3.1) we have  $c_1=0$ , and  $c_2 \neq 0$ . On the other hand, by integrating  $P_2(z) = (x^2 - a^2)/2$  ( $n-2$ )-times we obtain

$$P_n(z) = \frac{z^n}{n!} - \frac{a^2}{2} \frac{z^{n-2}}{(n-2)!} + \dots = \frac{1}{n!} (z^n - a^2 \frac{n(n-1)}{2} z^{n-2} + \dots )$$

and (10) shows that

$$(4.1) \quad \sum_1^n (z_j^{(n)})^2 = |\alpha|^2 n(n-1) .$$

This implies by (4.1), that

$$(R(P_n))^2 = \frac{1}{n} \sum_1^n |z_j^{(n)}|^2 \geq |\sum_1^n (z_j^{(n)})^2| = |\alpha|^2(n-1) ,$$

or

$$(4.2) \quad R(P_n) \geq |\alpha| \sqrt{n-1}, \quad (n \geq 2) .$$

This, however, is precisely (3.12) for  $k = 2$ .

If the zeros of all polynomials (3.1) are real, then we have an Appell sequence of the so-called Laguerre-Polya-Schur class (for references see [3]). In this case we have the equality sign in (4.2) for all  $n$ . An important example are the Hermite polynomials

$$H_n(z) = n! \sum_{r=0}^{[n/2]} \frac{(-1)^r}{r!} \frac{(2z)^{n-2r}}{(n-2r)!} , \quad (\text{see [4, page 105]}) ,$$

when (4.2) becomes the equation

$$R(H_n) = 2^{-\frac{1}{2}} \sqrt{n-1} .$$

This seems the place to mention the different behavior of the two quantities  $R(P_n)$

and

$$(4.3) \quad \max_j |z_j^{(n)}| .$$

Indeed, observe that we may express any Appell sequence (3.1) as

$$P_n(z) = \int_{z_1^{(n)}}^z dx_1 \int_{z_1^{(n-1)}}^{x_1} dx_2 \cdots \int_{z_1^{(1)}}^{x_{n-1}} dx_n .$$

Here

$$z_1^{(r)} \quad (r = 1, 2, \dots)$$

can be an arbitrarily chosen sequence of points of the complex plane. This implies that the sequence (4.3) may grow as fast as we wish as  $n \rightarrow \infty$ .

An open question: How fast can the quadratic radius  $R(P_n)$  grow as  $n \rightarrow \infty$ ?

5. Fred Sauer's numerical examples of Conjecture 1. As already mentioned Fred verified the inequality (7) in some 25 cases. We record here three of these examples:

| n | $P_n(z)$                      | $R(P_n)$ | $R(P_n')$ | $\sqrt{(n-2)/(n-1)} R(P_n)$ |
|---|-------------------------------|----------|-----------|-----------------------------|
| 5 | $z^5 + iz^2 + 5z - 2$         | 1.35871  | 1.16038   | 1.17668                     |
| 4 | $z^4 - iz^2 + z - i$          | 1.07753  | .68454    | .87980                      |
| 4 | $z^4 + (1+2i)z^2 - z + (1-i)$ | 1.27456  | .86366    | 1.04068                     |

The last two columns illustrate the inequality (7).

Appendix. A Conjecture 2 equivalent to Conjecture 1. Let

$$(A.1) \quad z_j = x_j + iy_j, \quad w_k = u_k + iv_k .$$

We rewrite Conjecture 1 replacing the inequality (7) by an obviously equivalent one

Conjecture 1. We have

$$(A.2) \quad \sum_1^n |z_j|^2 \geq \frac{n}{n-2} \sum_1^{n-1} |w_k|^2 .$$

with the equality sign iff the  $z_j$  are real, or on a straight line.

An equivalent conjecture is

Conjecture 2. We have

$$(A.3) \quad \sum_1^n x_j^2 \geq \frac{n}{n-2} \sum_1^{n-1} u_k^2 ,$$

with the equality sign iff the  $z_j$  are real, or on a straight line.

1. Conjecture 2 implies Conjecture 1. Applying Conjecture 2 in the two directions

Ox and Oy we obtain

$$(A.4) \quad \sum x_j^2 \geq \frac{n}{n-2} \sum u_k^2, \quad \sum y_j^2 \geq \frac{n}{n-2} \sum v_k^2$$

and adding them we obtain the inequality (A.2). Moreover, the equality sign in (A.2)

implies the equality in both relations (A.4), hence the  $z_j$  are real or on a line.

2. Conjecture 1 implies Conjecture 2. Indeed, by (11) from our Introduction we have,

using (A.1), the equation

$$\sum (x_j + iy_j)^2 = \frac{n}{n-2} \sum (u_k^2 + iv_k^2) .$$

Equating the real parts of both sides

$$\sum x_j^2 - \sum y_j^2 = \frac{n}{n-2} \sum u_k^2 - \frac{n}{n-2} \sum v_k^2 ,$$

whence

$$(A.5) \quad \sum x_j^2 - \frac{n}{n-2} \sum u_k^2 = \sum y_j^2 - \frac{n}{n-2} \sum v_k^2 (= \lambda) ,$$

$\lambda$  denoting the common value of both sides of (A.5).

From (A.5) we obtain

$$(A.6) \quad \sum |z_j|^2 - \frac{n}{n-2} \sum |w_k|^2 = 2\lambda .$$

Now (A.2) shows that  $\lambda \geq 0$ , and from (A.5) we see that the inequalities (A.4) hold.

Moreover, the equality sign in (A.2) shows that  $\lambda = 0$  in (A.5) and therefore we have

equality in (A.3) iff the  $z_j$  are real or rectilinear.

References

1. F. S. van den Berg, Nochmals über abgeleide Wurzelpunkte, Nieuw Archief voor Wiskunde, 15 (1888), 100-164.
2. I. J. Schoenberg, Mathematical Time Exposures. Math. Assoc. of America, 1982.
3. \_\_\_\_\_, On the spans of polynomials and the spans of a Laguerre-Polya-Schur sequence of polynomials. MRC Technical Summary Report #2757. Submitted to the Pacific J. of Math.
4. G. Szegő, Orthogonal Polynomials, American Math. Soc. Colloquium Publications, vol. XXIII, Revised Edition, 1959.

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| 20. ABSTRACT (Continue on reverse side if necessary and identify by block number)<br><br>Let<br>$P_n(z) = z^n + a_2 z^{n-2} + \dots + a_n = \prod_{j=1}^n (z - z_j)$<br>be a polynomial in $C[z]$ having the origin as the center of gravity of its zeros $z_j$ . We call<br>$R(P_n) = \left( \frac{1}{n} \sum_{j=1}^n  z_j ^2 \right)^{1/2}$ |                       |  |

ABSTRACT (continued)

the quadratic radius of  $P_n(z)$ . We also consider the derivative

$$P'_n(z) = nz^{n-1} + (n-2)a_2z^{n-3} + \dots + a_{n-1} = n \prod_{k=1}^{n-1} (z-w_k)$$

and its quadratic radius

$$R(P'_n) = \left( \frac{1}{n-1} \sum_{k=1}^{n-1} |w_k|^2 \right)^{1/2}.$$

The main purpose of this note is to state

Conjecture 1. We have the inequality

$$(1) \quad R(P'_n) \leq \sqrt{\frac{n-2}{n-1}} R(P_n),$$

with the equality sign if and only all the zeros  $z_j$  of  $P_n(z)$  are real, or equivalently, all  $z_j$  are on a straight line of  $\mathbf{C}$ .

We prove (1) for  $n = 3$ . Also for binomial polynomials of the form  $P_n(z) = z^n + a_k z^{n-k}$  ( $2 \leq k \leq n$ ). We prove directly other consequences of Conjecture 1.

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